

Sensitivity of Critical Transmission Ranges to Node Placement Distributions

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Abstract—We consider the geometric random graph where n points are distributed independently on the unit interval $[0, 1]$ according to some probability distribution function F with density f . Two nodes are adjacent (i.e., communicate with each other) if their distance is less than some transmission range. We survey results, some classical and some recently obtained by the authors, concerning the existence of zero-one laws for graph connectivity, the type of zero-one laws under the specific assumptions made, the form of its critical scaling and its dependence on the density f . We also present results and conjectures concerning the width of the corresponding phase transition. Engineering implications are discussed for power allocation.

Index Terms—Geometric random graphs, Connectivity, Zero-one laws, Phase transitions, Power allocation.

I. INTRODUCTION

OUR starting point is a paper by Gupta and Kumar [14] which has recently revived interest in the disk model as a framework for wireless ad-hoc networks. The setting is that of a wireless ad-hoc network serving n users (interchangeably referred to as nodes) which are distributed over some region \mathbb{D} of the plane. The nodes, labelled $1, 2, \dots, n$, are placed at the random locations $\mathbf{X}_1, \dots, \mathbf{X}_n$, respectively, in \mathbb{D} . A simplified pathloss model is assumed, and there is no user interference and no fading. Users all transmit at the same power level P . For distinct users i and j located at \mathbf{X}_i and \mathbf{X}_j , their received power $P_{i,j}$ is given by

$$P_{i,j} := P \cdot \|\mathbf{X}_i - \mathbf{X}_j\|^{-\nu}$$

for some pathloss exponent $\nu > 0$. Under this communication model, nodes i and j are then able to communicate if $P_{i,j} \geq \Gamma$ for some threshold $\Gamma > 0$ (whose selection is guided by

bit error rate considerations, among other things). As this condition is equivalent to requiring

$$\|\mathbf{X}_i - \mathbf{X}_j\| \leq \rho \quad \text{with} \quad \rho := \left(\frac{P}{\Gamma}\right)^{1/\nu}, \quad (1)$$

we can view the transmission range ρ as a proxy for the transmit power P .

Given a transmission range $\rho > 0$, the relation (1) defines a notion of adjacency amongst nodes, giving rise to the undirected geometric random graph $\mathbb{G}(n; \rho)$ on the set of nodes $1, \dots, n$. Thus, the presence of an edge between two nodes captures their ability to communicate directly and reliably with each other. However, viewed as systems, networks are “greater than the sum of their parts” and “network connectivity” emerges from one-hop connectivity as network resources collectively enable end-to-end data transfer between participating nodes. Under the underlying assumptions of the disk model it is customary to identify network connectivity with the usual notion of *graph connectivity* in $\mathbb{G}(n; \rho)$ (whereby every pair of nodes can be linked by at least one path over the edges of the graph).

A. Critical power levels for network connectivity

In the context of this disk model, a natural question consists in determining the minimum power level P_n needed to ensure (network) connectivity amongst the nodes located at $\mathbf{X}_1, \dots, \mathbf{X}_n$ – This was in fact the very problem considered by Gupta and Kumar in [14]. Expressed in terms of the communication range, this amounts to considering the *critical transmission range* R_n defined by

$$R_n := \min(\rho > 0 : \mathbb{G}(n; \rho) \text{ is connected}), \quad (2)$$

and the minimum power level P_n is then simply given by

$$P_n = \Gamma R_n^\nu. \quad (3)$$

The quantity R_n (hence P_n) being a function of $\mathbf{X}_1, \dots, \mathbf{X}_n$, it has limited operational use since node locations are neither available, nor should their knowledge be expected, especially in the presence of node mobility. To make matters worse, the probability distribution function of the rv R_n given by

$$P(n; \rho) := \mathbb{P}[R_n \leq \rho], \quad \rho > 0$$

is typically not available in closed form. To the best of our knowledge the only situation where such a closed form has been obtained is the one-dimensional network discussed in Section IV; see [21] for details. Even there the expression

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yields no insights on how the statistics of $\mathbf{X}_1, \dots, \mathbf{X}_n$ affect graph connectivity.

Fortunately a case can be made that efficient power allocation matters only when dealing with a large number of users. After all this is a regime where the problem assumes added relevance (as well as some urgency) since energy resources are painfully finite. In that asymptotic regime it is hoped that limiting results would be available for R_n , leading to a reasonably good approximation to it by a *non-random* quantity ρ_n^* , say

$$R_n \simeq \rho_n^* \quad \text{with very high probability.} \quad (4)$$

A possible formalization of this idea is provided by the convergence¹

$$\frac{R_n}{\rho_n^*} \xrightarrow{P} 1. \quad (5)$$

This result immediately suggests an approximation to the minimum power level P_n by means of the *non-random* $\pi_n^* := \Gamma(\rho_n^*)^\nu$.

Such developments are indeed possible under appropriate assumptions; see below. The relevant results have been obtained from several complementary viewpoints which can be reconciled upon noting that $\mathbb{G}(n; \rho)$ is connected if and only if $R_n \leq \rho$, so that

$$\mathbb{P}[\mathbb{G}(n; \rho) \text{ is connected}] = P(n; \rho), \quad \rho > 0.$$

The validity of (4)-(5) is then seen to be equivalent to the *zero-one law*

$$\lim_{n \rightarrow \infty} P(n; \rho_n) = 0 \quad \text{if } \rho_n \text{ “(much) smaller than” } \rho_n^*$$

and

$$\lim_{n \rightarrow \infty} P(n; \rho_n) = 1 \quad \text{if } \rho_n \text{ “(much) larger than” } \rho_n^*.$$

The approximation ρ_n^* to the critical transmission range R_n acts as a boundary in the space of scalings, and is often referred to as a *critical scaling*.

B. Classical results

When studying these zero-one laws the situation most often considered is the one where the locations $\mathbf{X}_1, \dots, \mathbf{X}_n$ are independent and uniformly distributed over the domain \mathbb{D} . With the transmission range scaled as a function of n according to

$$\pi \rho_n^2 = \frac{\log n + \alpha_n}{n}, \quad n = 1, 2, \dots \quad (6)$$

for some sequence $\alpha : \mathbb{N}_0 \rightarrow \mathbb{R}$, several authors [14], [32] have shown that

$$\lim_{n \rightarrow \infty} P(n; \rho_n) = 0 \quad \text{if } \lim_{n \rightarrow \infty} \alpha_n = -\infty \quad (7)$$

and

$$\lim_{n \rightarrow \infty} P(n; \rho_n) = 1 \quad \text{if } \lim_{n \rightarrow \infty} \alpha_n = \infty. \quad (8)$$

In particular, with

$$\pi \rho_n^2 \sim c \frac{\log n}{n},$$

¹See Section I-D for notation and conventions.

the zero-one law (7)-(8) implies

$$\lim_{n \rightarrow \infty} P(n; \rho_n) = \begin{cases} 0 & \text{if } 0 < c < 1 \\ 1 & \text{if } 1 < c, \end{cases} \quad (9)$$

and the critical scaling is then determined by

$$\pi \rho_n^2 = \frac{\log n}{n}, \quad n = 1, 2, \dots \quad (10)$$

In light of these results, a natural question arises as to their *dependence* (and therefore *sensitivity*) with respect to the node placement distribution. Non-uniform node placement naturally occurs when considering node mobility, e.g., random waypoint mobility [8], [35], [36]. In [31] Penrose gave a partial answer when the locations $\mathbf{X}_1, \dots, \mathbf{X}_n$ are i.i.d. rvs distributed over the domain \mathbb{D} according to some probability distribution F with density f . Under mild continuity assumptions on f , Penrose showed [31, Thm. 1.1] that (9) still holds if the transmission range is scaled according to

$$\pi \rho_n^2 \sim c \frac{1}{M(F)} \cdot \frac{\log n}{n} \quad (11)$$

with the constant $M(F)$ determined by the minima of f on \mathbb{D} and on its boundary. As a result the critical scaling is now determined through

$$\pi \rho_n^2 = \frac{1}{M(F)} \cdot \frac{\log n}{n}, \quad n = 1, 2, \dots \quad (12)$$

C. Contributions

There remains open the question as to what is the analog of the zero-one law (7)-(8) under *non-uniform* node placement distributions. Interest in such results stems from the fact that they express *extreme* sensitivity to deviations from a critical scaling, and suggest the likely presence of sharp phase transitions with possible implications for power allocation. To the best of our knowledge, no results have been reported on the analogs of (7)-(8) in the non-uniform setting.

In this paper we survey available results concerning this issue in the context of *one-dimensional* networks where n points are distributed independently on the unit interval $[0, 1]$ according to some probability distribution function F with density f . Under various sets of assumptions on f , we discuss (i) the *existence* of zero-one laws for graph connectivity, (ii) the *type* of the zero-one laws available under the specific assumptions made, (iii) the *form* of the corresponding critical scaling and (iv) its *dependence* on the density function f used for node placement. Where appropriate, we also present results and conjectures concerning the width of the corresponding phase transition.

As part of this narrative we develop a single *unifying* framework to present, compare and contrast the surveyed results, some classical and some recently obtained by the authors. In particular we approach the convergence (5) (and related results) through the asymptotic properties of maximal spacings induced by i.i.d. variates on the unit interval [33]. This approach leads naturally to the notions of weak, strong and very strong zero-one laws, with attending critical thresholds; this classification is at the heart of some of our conclusions. Proofs have been omitted due to space limitations but can be

found in the Ph.D. thesis [15] and in the papers [18], [20], [21] and [22]. We refer the reader to these sources for any additional information and for a detailed discussion of the finer technical points.

Critical scalings and sharp phase transitions will be shown to exist in many situations, and this is certainly rather pleasing from a mathematical standpoint. Ideally, one envisions such results helping guide the determination of reasonably efficient (if not minimum) power levels which guarantee connectivity. However, large scale wireless ad-hoc networks are expected to be deployed in many vastly different environments with large variations in critical system parameters. Sound engineering practice requires that performance should not depend too heavily on model parameters which are either unrealistic or hard to obtain. These concerns point to the need to better understand the robustness of the aforementioned mathematical results to parameter variations.

To that end we provide a systematic discussion of critical scalings under various assumptions on the density function f . In the non-vanishing case (i.e., $f_* > 0$ with f_* the minimum of f), the critical scaling depends on the *inverse* of f_* . As it is usually very difficult to estimate f_* accurately, this eventually leads to adopting power allocations far more conservative than the ones suggested by either the strong or the very strong zero-one laws: Only weak zero-one laws turn out to be operationally relevant for they require no information regarding the density f . As a result, the selected transmission ranges are orders of magnitude larger than $\frac{\log n}{n}$, the critical scaling associated with uniform node placement on the unit interval; see details in Sections VI and VII. This conservative approach is further amplified when the density for node placement vanishes (i.e., $f_* = 0$). As a rule, the critical scaling being very sensitive to f_* , only the weak zero-one laws can be leveraged in any practical sense!

One-dimensional random graphs are of interest in their own right as simple models of wireless ad-hoc networks constrained over “linear” highways. They have been discussed by a number of authors mostly under uniform node placement, e.g., see [5], [7], [8], [9], [11], [13], [17], [18], [21], [28], [29], [34], [35], [36] (and references therein). Here we have elected to discuss one-dimensional networks (rather than their more physically relevant two-dimensional counterparts) because a fairly complete picture of its zero-one laws is now available. This is a consequence of the fact that key one-dimensional results flow from properties of maximal spacings, thereby avoiding many of the technical difficulties associated with higher-dimensional geometry, e.g., see [30], [31], [32]. vs. [18], [20], [21].

Usually, whenever a one-dimensional result has a higher-dimensional counterpart, they are structurally similar, e.g., compare (6)–(8) vs. Theorem 4.3, or (11) vs. Theorem 5.2. We expect that this similarity will continue to hold for one-dimensional results whose analogs in higher dimensions are not yet known. Thus, the conclusions reached earlier concerning the lack of engineering relevance of strong and very strong zero-one laws should hold irrespectively of dimension. The one-dimensional model, through the present survey, helps make the case although some of the “evidence” may not be in yet for the original disk model. The higher-dimensional case

is technically more involved and is not completely understood as of this writing. We hope that the discussion given here will stimulate work along these lines.

The paper is organized as follows: In Section II we present the one-dimensional model with its basic assumptions. The critical transmission range is then related to the maximal spacings in Section III-A. Zero-one laws are introduced in Section III-B where they are characterized in terms of properties of maximal spacings. The uniform case is discussed in Section IV: The zero-one laws are discussed in Section IV-A and followed in Section IV-B by results on the width of the phase transition. In Section V we cover the non-uniform case when the density does not vanish. Section VI contains a discussion of some of the implications of these results. The case when the density vanishes is discussed in Section VII.

D. Notation and conventions

Statements involving limits, including asymptotic equivalences, are always understood with n going to infinity. Almost everywhere is abbreviated as a.e. and all such statements are made with respect to Lebesgue measure λ on the unit interval $[0, 1]$. The random variables (rvs) under consideration are all defined on the same probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. All probabilistic statements are made with respect to this probability measure \mathbb{P} , and we denote the corresponding expectation operator by \mathbb{E} . The notation \xrightarrow{P}_n (resp. \Rightarrow_n) signifies convergence in probability (resp. convergence in distribution) with n going to infinity. Also, we use the notation $=_{st}$ to indicate distributional equality. The indicator function of an event E is simply denoted by $1[E]$.

II. THE ONE-DIMENSIONAL MODEL

Throughout, let $\{X_i, i = 1, 2, \dots\}$ denote a sequence of i.i.d. rvs which are distributed on the unit interval $[0, 1]$ according to some common probability distribution function F . For each $n = 2, 3, \dots$, we think of X_1, \dots, X_n as the locations of n nodes, labelled $1, \dots, n$, in the interval $[0, 1]$. Given a fixed transmission range $\rho > 0$, nodes i and j are said to be adjacent if $|X_i - X_j| \leq \rho$, in which case an undirected edge exists between these two nodes. The geometric random graph induced by this notion of adjacency is denoted by $\mathbb{G}(n; \rho)$. Again, the probability that the graph $\mathbb{G}(n; \rho)$ is connected, is given by

$$P(n; \rho) = \mathbb{P}[\mathbb{G}(n; \rho) \text{ is connected}]; \quad (13)$$

obviously $P(n; \rho) = 1$ whenever $\rho \geq 1$.

This one-dimensional model arises in the same manner as the two-dimensional disk model: The users all transmit at the same power level P under a simplified pathloss, no user interference and no fading. For distinct users i and j located at X_i and X_j , their received power $P_{i,j}$ is given by

$$P_{i,j} := P \cdot |X_i - X_j|^{-\nu}$$

for some $\nu > 0$. Nodes i and j are then able to communicate if $P_{i,j} \geq \Gamma$ for some threshold $\Gamma > 0$, a condition equivalent to

$$|X_i - X_j| \leq \rho \quad \text{with} \quad \rho := \left(\frac{P}{\Gamma}\right)^{1/\nu}.$$

A number of assumptions are imposed on F with the most basic one being given first.

Assumption 1: The distribution $F : [0, 1] \rightarrow [0, 1]$ is absolutely continuous (with respect to λ).

Thus, F is differentiable a.e. on $[0, 1]$ with $F(0) = 0$ and $F(1) = 1$, and the relation

$$F(x) = \int_0^x f(t)dt, \quad x \in [0, 1] \quad (14)$$

holds for some density function $f : [0, 1] \rightarrow \mathbb{R}_+$. This density f is determined up to a.e. equivalence [37, Section 9.2].

The essential infimum²

$$f_\star := \text{ess inf } (f(x), x \in [0, 1])$$

is uniquely determined by F , hence by (the equivalence class of) f . There is no loss of generality in selecting (as we do from now on) the density f appearing in (14) so that

$$f_\star = \inf (f(x), x \in [0, 1]). \quad (15)$$

This can be achieved by suitably redefining f on a set of zero Lebesgue measure, and will not affect the results obtained here since this procedure leaves F unchanged.

It is plain that $0 \leq f_\star \leq 1$ with $f_\star = 1$ corresponding to F being the uniform distribution. Most of our results require the density f to be bounded away from zero in the following technical sense.

Assumption 2: With the density f selected such that (15) holds, there exists x_\star in the interval $[0, 1]$ such that

$$f_\star = f(x_\star) > 0, \quad (16)$$

and this point x_\star is a point of continuity for f .

The minimizer appearing in Assumption 2 is not necessarily unique. Additional assumptions will be made in due course as needed.

III. CRITICAL TRANSMISSION RANGES, MAXIMAL SPACINGS AND ZERO-ONE LAWS

Fix $n = 2, 3, \dots$. As before the minimum power level P_n needed to ensure network connectivity amongst the nodes located at X_1, \dots, X_n is related to the critical transmission range R_n through the relation (3) with R_n given by (2).

A. Maximal spacings

With the node locations X_1, \dots, X_n , we associate rvs $X_{n,1}, \dots, X_{n,n}$ which are the locations of the n users arranged in increasing order, i.e., $X_{n,1} \leq \dots \leq X_{n,n}$. The rvs $X_{n,1}, \dots, X_{n,n}$ are the *order statistics* [3] associated with the n i.i.d. rvs X_1, \dots, X_n . With the convention $X_{n,0} = 0$ and $X_{n,n+1} = 1$, we define the spacing rvs

$$L_{n,k} := X_{n,k} - X_{n,k-1}, \quad k = 1, \dots, n+1 \quad (17)$$

and the *maximal spacing* M_n is the rv given by

$$M_n := \max (L_{n,k}, k = 2, \dots, n). \quad (18)$$

²Recall that

$$f_\star = \sup (a \in \mathbb{R} : \lambda(\{x \in [0, 1] : f(x) < a\}) = 0).$$

We obviously have

$$R_n = M_n. \quad (19)$$

For each $\rho > 0$, the graph $\mathbb{G}(n; \rho)$ is connected if and only if $M_n \leq \rho$, so that (13) becomes

$$P(n; \rho) = \mathbb{P}[M_n \leq \rho]. \quad (20)$$

We shall scale the transmission range with the number of nodes in the network through mappings $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$; we refer to any such mapping as a *scaling*. With the help of the relationship (3) we pass from a scaling $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ on the transmission range to the corresponding scaling $\pi : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ on the power level by setting

$$\pi_n = \Gamma(\rho_n)^\nu, \quad n = 1, 2, \dots \quad (21)$$

All results will be given in terms of the transmission range with an obvious translation to scalings for the power level via (21).

B. Zero-one laws

We adopt the following terminology regarding zero-one laws for the property of graph connectivity in $\mathbb{G}(n; \rho)$ [28, p. 376]: A *strong* zero-one law is said to hold with critical scaling $\rho^\star : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ if for any scaling $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ satisfying

$$\lim_{n \rightarrow \infty} \frac{\rho_n}{\rho_n^\star} = c \quad (22)$$

for some $c > 0$, we have

$$\lim_{n \rightarrow \infty} P(n; \rho_n) = \begin{cases} 0 & \text{if } 0 < c < 1 \\ 1 & \text{if } 1 < c. \end{cases} \quad (23)$$

Any scaling $\rho^\star : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ appearing in (22)-(23) will be called a *strong* critical scaling.

We have the following simple characterization in terms of maximal spacings.

Proposition 3.1: The property of graph connectivity in $\mathbb{G}(n; \rho)$ admits a strong zero-one law with critical scaling $\rho^\star : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ if and only if

$$\frac{M_n}{\rho_n^\star} \xrightarrow{P} 1. \quad (24)$$

This equivalence can be easily understood from the following heuristic argument: Imagine you wish to test whether the scaling $\rho^\star : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ is a strong critical scaling for the property of graph connectivity. A natural way to do so consists in picking a scaling $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ satisfying (22) and checking whether (23) holds. We then observe from (20) that

$$P(n; \rho_n) = \mathbb{P}[M_n \leq \rho_n] = \mathbb{P}\left[\frac{M_n}{\rho_n^\star} \leq \frac{\rho_n}{\rho_n^\star}\right] \quad (25)$$

for all $n = 1, 2, \dots$, so that

$$P(n; \rho_n) \simeq \mathbb{P}\left[\frac{M_n}{\rho_n^\star} \leq c\right] \quad (26)$$

for large n . It is now plain from (26) that $\frac{M_n}{\rho_n^\star}$ “stabilizing” in the sense of (24) is equivalent to the validity of a strong zero-one law; see [20] for details.

During the discussion we shall also have use for the following notion [28, p. 376]: A *weak zero-one law* is said to hold with critical scaling $\rho^* : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ if

$$\lim_{n \rightarrow \infty} P(n; \rho_n) = \begin{cases} 0 & \text{if } \lim_{n \rightarrow \infty} \frac{\rho_n}{\rho_n^*} = 0 \\ 1 & \text{if } \lim_{n \rightarrow \infty} \frac{\rho_n}{\rho_n^*} = \infty \end{cases} \quad (27)$$

with scaling $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$. Any scaling $\rho^* : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ appearing in (27) will be called a *weak critical scaling*.

C. Comments and useful facts

In its weak form the one law (resp. zero law) emerges when considering scalings $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ which are *at least* an order of magnitude larger (resp. smaller) than ρ^* . On the other hand, under the strong law, for n sufficiently large, a transmission range ρ_n suitably larger (resp. smaller) than ρ_n^* ensures $P(n; \rho_n) \simeq 1$ (resp. $P(n; \rho_n) \simeq 0$) provided $\rho_n \sim c\rho_n^*$ with $c > 1$ (resp. $0 < c < 1$). This is in sharp contrast with (27) in that the strong one law (resp. zero law) still occurs with scalings $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ which are larger (resp. smaller) than ρ^* but of the *same* order of magnitude as ρ^* !

A convenient way to formulate these differences is to observe that the two types of zero-one laws deal with scalings $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ for which the limit

$$\lim_{n \rightarrow \infty} \frac{\rho_n}{\rho_n^*} = c$$

exists for some c in $[0, \infty]$. The strong zero-one law corresponds to c in $(0, \infty)$ while the weak zero-one law allows only $c = 0, \infty$. The terminology now becomes clear: The weak zero-one law requires a “more brutal” scale separation from the critical scaling $\rho^* : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ than the strong zero-one law in order to ensure either $P(n; \rho_n) \simeq 0$ or $P(n; \rho_n) \simeq 1$. A simple monotonicity argument (in ρ) shows that a strong zero-one law is necessarily a weak zero-one law, and a strong critical scaling is therefore also a weak critical scaling.

Critical scalings are *not* unique as they emerge from limiting properties. Thus, consider two strong critical scalings $\rho^*, \rho^{**} : \mathbb{N}_0 \rightarrow \mathbb{R}_+$. It follows from Proposition 3.1 that they are necessarily *asymptotically equivalent*, i.e., $\rho_n^* \sim \rho_n^{**}$. On the other hand, if the scaling ρ^* is a weak critical scaling, then so will the scaling ρ^{**} be provided they are *order equivalent*, i.e.,

$$0 < \liminf_{n \rightarrow \infty} \frac{\rho_n^*}{\rho_n^{**}} \leq \limsup_{n \rightarrow \infty} \frac{\rho_n^*}{\rho_n^{**}} < \infty.$$

IV. THE UNIFORM CASE

We begin with the well-studied case when F is the uniform distribution on $[0, 1]$, namely

$$F(x) = x, \quad x \in [0, 1].$$

The density function is then constant with $f(x) = 1$ on the interval $[0, 1]$, and $f_* = 1$. Assumptions 1 and 2 obviously hold in that case.

As will become apparent from the discussion below, the scaling $\rho_U^* : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ given by

$$\rho_{U,n}^* = \frac{\log n}{n}, \quad n = 1, 2, \dots \quad (28)$$

occupies a singular place in the space of scalings. The first indication of this special status can be found in the classical convergence results given next. They were originally obtained by Lévy [27] via geometric arguments, but have been rederived by Darling [2] (by analytical techniques), and others; see Devroye’s paper [6] for additional references. A simple proof was also given in [21].

In order to state the results compactly, let Λ denote any \mathbb{R} -valued rv with probability distribution given by

$$\mathbb{P}[\Lambda \leq x] = g(x) := e^{-e^{-x}}, \quad x \in \mathbb{R}. \quad (29)$$

Any rv distributed according to (29) is called a Gumbel rv.

Theorem 4.1: Under the uniform distribution, we have

$$\frac{M_n}{\rho_{U,n}^*} \xrightarrow{P} 1 \quad (30)$$

and

$$nM_n - \log n \Rightarrow_n \Lambda. \quad (31)$$

It is now a simple matter to see from (20) and (31) that

$$\lim_{n \rightarrow \infty} P\left(n; \frac{\log n + x}{n}\right) = e^{-e^{-x}}, \quad x \in \mathbb{R}. \quad (32)$$

All the results given next in the uniform case are in fact consequences of (30) and (31) (via (32)).

A. Zero-one laws

Combining Theorem 4.1 with Proposition 3.1 we obtain the following strong zero-one law for the uniform case.

Theorem 4.2: Assume F to be the uniform distribution. For any scaling $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ such that

$$\lim_{n \rightarrow \infty} \frac{\rho_n}{\rho_{U,n}^*} = c \quad (33)$$

for some $c > 0$, we have

$$\lim_{n \rightarrow \infty} P(n; \rho_n) = \begin{cases} 0 & \text{if } 0 < c < 1 \\ 1 & \text{if } 1 < c. \end{cases} \quad (34)$$

Thus, the scaling ρ_U^* is a strong critical scaling in the uniform case. This zero-one law is already contained in Theorem 1 by Appel and Russo [1, p. 352]. More recently, Muthukrishnan and Pandurangan [29, Thm. 2.2] have also derived (33)-(34) by a bin-covering technique.

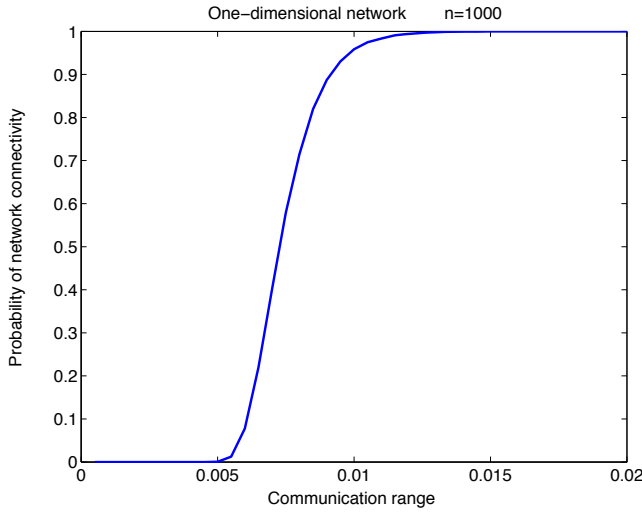
Next, in anticipation to strengthening Theorem 4.2, we note that there is no loss of generality in writing any scaling $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ in the form

$$\begin{aligned} \rho_n &= \frac{1}{n} (\log n + \alpha_n) \\ &= \rho_{U,n}^* + \frac{\alpha_n}{n}, \quad n = 1, 2, \dots \end{aligned} \quad (35)$$

for some function $\alpha : \mathbb{N}_0 \rightarrow \mathbb{R}$, hereafter referred to as a deviation function.

Theorem 4.3: Assume F to be the uniform distribution. For any scaling $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ written in the form (35), it holds that

$$\lim_{n \rightarrow \infty} P(n; \rho_n) = \begin{cases} 0 & \text{if } \lim_{n \rightarrow \infty} \alpha_n = -\infty \\ 1 & \text{if } \lim_{n \rightarrow \infty} \alpha_n = +\infty. \end{cases} \quad (36)$$

Fig. 1. Sharp phase transition with $n = 1,000$

In [18], we derived Theorem 4.3 by applying the method of first and second moments [24, p. 55] to the number of breakpoint users in $\mathbb{G}(n; \rho)$.³ In [21] we provided a second and shorter proof based on (32) and on a monotonicity (in ρ) argument.

From Theorem 4.3 we see that a perturbation α from the critical scaling yields the one-law (resp. zero-law) if $\lim_{n \rightarrow \infty} \alpha_n = \infty$ (resp. $\lim_{n \rightarrow \infty} \alpha_n = -\infty$) with *no additional constraint* on α . Contrast this with (34) where we allow only scalings of the form $\rho_n \sim c\rho_{U,n}^*$ with $c > 0$ and $c \neq 1$, so that $\alpha_n \sim (c-1)\log n$. It is now plain that (34) is indeed implied by (36). Whereas “small” deviations of the form $\alpha_n = \pm \log \log n$ are covered by Theorem 4.3, they are not covered by the zero-one law (34) (since $\alpha_n = o(\log n)$ with $c = 0$). In the context of Theorem 4.2 the more delicate boundary case $c = 1$ is partially handled with the help of Theorem 4.3. For these reasons it is most appropriate to call (35)-(36) a *very strong* (and not merely a strong) zero-one law for the property of graph connectivity; accordingly we refer to the scaling ρ_U^* as a *very strong critical scaling*.

B. Transition widths and phase transitions

The “sensitivity” to small deviations implied by Theorem 4.3 is in line with the very sharp phase transition already apparent from graphs reported in several papers, e.g., see [9], [13], [25] and [26], and from Figure 1 below. Results formalizing the sharpness of this transition have been obtained recently in [12], [16], [17] and [21], and are now summarized.

For each $n = 2, 3, \dots$, the mapping $\rho \rightarrow P(n; \rho)$ is continuous and strictly monotone increasing on $[0, 1]$. Given p in $(0, 1)$, these properties guarantee the existence and uniqueness of solutions to the equation

$$P(n; \rho) = p, \quad \rho \in (0, 1). \quad (37)$$

Let $\rho_{U,n}(p)$ denote this unique solution. Its behavior for large n is given next.

³For each $k = 1, \dots, n$, node k is said to be a *breakpoint* node in the random graph $\mathbb{G}(n; \rho)$ if the interval $[X_k, X_k + \rho]$ does *not* contain any other node of the graph.

Theorem 4.4: For every p in the interval $(0, 1)$, we have

$$\begin{aligned} \rho_{U,n}(p) &= \frac{\log n}{n} - \frac{1}{n} \log \left(\log \left(\frac{1}{p} \right) \right) + o(n^{-1}) \\ &= \rho_{U,n}^* - \frac{1}{n} \log \left(\log \left(\frac{1}{p} \right) \right) + o(n^{-1}). \end{aligned} \quad (38)$$

Theorem 4.4 can be argued as follows: For each x in \mathbb{R} , the convergence (32) yields the approximation

$$P \left(n; \frac{\log n + x}{n} \right) \simeq g(x) \quad (39)$$

for large enough n (with $g(x)$ as defined at (29)). The mapping $g : \mathbb{R} \rightarrow \mathbb{R}_+ : x \rightarrow g(x)$ is strictly monotone and continuous with $\lim_{x \rightarrow -\infty} g(x) = 0$ and $\lim_{x \rightarrow \infty} g(x) = 1$. Therefore, for each p in the interval $(0, 1)$, there exists a unique scalar x_p such that $g(x_p) = p$, namely

$$x_p = -\log(-\log p) = -\log \left(\log \left(\frac{1}{p} \right) \right). \quad (40)$$

Given p in the interval $(0, 1)$, the approximation (39) (with $x = x_p$) becomes

$$P \left(n; \rho_{U,n}^* + \frac{x_p}{n} \right) \simeq p$$

for large n , while the definition of $\rho_{U,n}(p)$ gives

$$P(n; \rho_{U,n}(p)) = p, \quad n = 2, 3, \dots$$

Combining these last two facts we conclude that

$$P \left(n; \rho_{U,n}^* + \frac{x_p}{n} \right) \simeq P(n; \rho_{U,n}(p))$$

for large n . Continuity suggests that $\rho_{U,n}^* + \frac{x_p}{n}$ and $\rho_{U,n}(p)$ behave in tandem asymptotically, and this lays the ground for the validity of (38); details are available in [21].⁴

Next, set

$$\delta_{U,n}(p) := \rho_{U,n}(1-p) - \rho_{U,n}(p), \quad p \in \left(0, \frac{1}{2}\right).$$

The transition width $\delta_{U,n}(p)$ measures the increase in transmission range needed in the n node network to drive the probability of connectivity from level p to level $1-p$. The more rapidly $\rho_{U,n}(p)$ decays as a function of n , the sharper the phase transition. The following result is an easy corollary to Theorem 4.4.

Corollary 4.5: For every p in the interval $(0, \frac{1}{2})$, we have

$$\delta_{U,n}(p) = \frac{C(p)}{n} + o(n^{-1}) \quad (41)$$

with constant $C(p)$ given by

$$C(p) := \log \left(\frac{\log p}{\log(1-p)} \right). \quad (42)$$

Recently Goel et al. [12, Thm. 1.1] have shown that

$$\delta_{U,n}(p) = O \left(\sqrt{\frac{-\log p}{n}} \right). \quad (43)$$

⁴Similar arguments can be made in the two-dimensional case on the basis of the analog of (32). See the conference papers [16] [17] for details.

In fact these *asymptotic bounds* were established for every *monotone* graph property. The results obtained here markedly improve on (43) in that *exact* asymptotics are now provided and the rate of decay (namely, n^{-1}) is much faster than the rough asymptotic bound given by (43). However, these conclusions hold only for graph connectivity.

As we close Section IV, a natural question arises as to what happens to these results when F is not the uniform distribution. This is taken on in Sections V and VII.

V. THE NON-UNIFORM CASE WITH $f_\star > 0$

A. The strong zero-one law

Under Assumptions 1 and 2, the scaling $\rho_F^\star : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ given by

$$\rho_{F,n}^\star := \frac{1}{f_\star} \cdot \frac{\log n}{n} = \frac{1}{f_\star} \cdot \rho_{U,n}^\star, \quad n = 1, 2, \dots \quad (44)$$

is well defined; in the uniform case it reduces to the critical scaling (28). The following result was established in [20] and constitutes the appropriate extension of Lévy's result (30) to non-uniform distributions.

Theorem 5.1: Under any distribution F satisfying Assumptions 1 and 2, we have

$$\frac{M_n}{\rho_{F,n}^\star} \xrightarrow{P} n. \quad (45)$$

With the help of Proposition 3.1 we see from (45) that Theorem 4.2 has the following analog in the non-uniform case.

Theorem 5.2: Assumptions 1 and 2 are enforced on the distribution F . For any scaling $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ such that

$$\lim_{n \rightarrow \infty} \frac{\rho_n}{\rho_{F,n}^\star} = c \quad (46)$$

for some $c > 0$, we have

$$\lim_{n \rightarrow \infty} P(n; \rho_n) = \begin{cases} 0 & \text{if } 0 < c < 1 \\ 1 & \text{if } 1 < c. \end{cases} \quad (47)$$

Thus, the scaling $\rho_F^\star : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ is a strong critical scaling for graph connectivity – Note the structural similarity with (11)-(12) in the higher dimensional case. Note also that f_\star is the only artifact of the density function which enters its definition – The actual location x_\star where the minimum is achieved plays no role as long as it is a point of continuity for f .

The convergence (45) is compatible with a multi-dimensional result obtained by Penrose [30]: Formally setting $d = 1$ in Theorem 1.1 of [30, p. 247] (discussed under the dimensional assumption $d \geq 2$), we obtain (45) in a.s. form.

B. Additional assumptions

A number of additional assumptions are needed to formulate the analog of Theorem 4.3 for non-uniform distributions.

Assumption 3: The distribution F admits a density function $f : [0, 1] \rightarrow \mathbb{R}_+$ which is continuous on the interval $[0, 1]$ except possibly at a finite number of points. Each point of

*discontinuity of f is either a removable discontinuity or a discontinuity of the first kind.*⁵

Under Assumption 3, there is no loss of generality in assuming (as we do from now on) that the density f is right-continuous with left limit at every point of discontinuity in the open interval $(0, 1)$, and continuous at the boundary points $x = 0, 1$. This can be achieved by suitably redefining f at the points of discontinuity and will not affect the results obtained here since this procedure leaves F unchanged.

With f_\star still defined by (15), we write

$$f^\star := \sup (f(x), x \in [0, 1]). \quad (48)$$

Through the possible redefinition mentioned above, Assumption 3 on f guarantees $0 \leq f_\star \leq f^\star < \infty$ with $f_\star \leq 1$.

Our next assumption constitutes a stronger form of Assumption 2.

Assumption 4: With the density f as selected in Assumption 3, there exists a single element x_\star in the interval $[0, 1]$ such that

$$f_\star = f(x_\star) > 0, \quad (49)$$

and this point x_\star is a point of continuity for f .

Assumptions 3 and 4 are strengthened versions of Assumptions 1 and 2, respectively. In particular, according to Assumption 4 the density function f is required to exhibit a *single* global minimum. We complement this property by requiring a representation for the density f which in effect imposes a form of smoothness near the minimizer x_\star .

Assumption 5: With the density f as selected in Assumption 3 and the unique minimizer x_\star as specified in Assumption 4, the density function f can be represented in the form

$$f(x) = f_\star + a|x - x_\star|^r + h(x), \quad x \in [0, 1] \quad (50)$$

for some parameters $r > 0$ and $a > 0$, and for some function $h : [0, 1] \rightarrow \mathbb{R}$ such that

$$\lim_{x \rightarrow x_\star} \frac{h(x)}{|x - x_\star|^r} = 0. \quad (51)$$

The conditions (50) and (51) are not overly restrictive. For instance, they hold when the density function f admits $2\ell + 1$ bounded derivatives $f^{(1)}, \dots, f^{(2\ell+1)} : [0, 1] \rightarrow \mathbb{R}$ such that

$$f^{(1)}(x_\star) = \dots = f^{(2\ell-1)}(x_\star) = 0, \quad f^{(2\ell)}(x_\star) > 0$$

for some positive integer ℓ when x_\star is a unique global minimum for f in the open interval $(0, 1)$. In that case, the existence of a Taylor series expansion at $x = x_\star$ leads to taking $r = 2\ell$ and

$$a = \frac{1}{(2\ell)!} f^{(2\ell)}(x_\star)$$

so that (51) holds with the choice

$$h(x) = f(x) - f(x_\star) - a(x - x_\star)^{2\ell}, \quad x \in [0, 1].$$

The conditions imposed on f may not be the weakest possible to guarantee the results. However, they cover most situations likely to be encountered in applications such as wireless networking.

⁵ A discontinuity of the first kind is also known as a jump discontinuity, and is characterized by the existence of right and left limits.

C. The very strong zero-one law

With Assumptions 3-5 enforced on F , introduce the scaling $\rho_F^{**} : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ given by

$$\rho_{F,n}^{**} := \frac{1}{f_*} \cdot \frac{1}{n} \left(\log n - \frac{1}{r} \log \log n \right) \quad (52)$$

for all $n = 1, 2, \dots$. This scaling reduces to the critical scaling (28) found in the uniform case (where $f_* = 1$ and $r = \infty$).

The results will assume a more symmetric form if we write a scaling $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ in the form

$$\begin{aligned} \rho_n &= \frac{1}{f_*} \cdot \frac{1}{n} \left(\log n - \frac{1}{r} \log \log n + \alpha_n \right) \\ &= \rho_{F,n}^{**} + \frac{1}{f_*} \frac{\alpha_n}{n}, \quad n = 1, 2, \dots \end{aligned} \quad (53)$$

for some deviation function $\alpha : \mathbb{N}_0 \rightarrow \mathbb{R}$. Again there is no loss of generality in using the representation (53). The analog of Theorem 4.3 for non-uniform distributions is presented next.

Theorem 5.3: *Assumptions 3-5 are enforced on the distribution F . Then, for any scaling $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ written in the form (53) with deviation function $\alpha : \mathbb{N}_0 \rightarrow \mathbb{R}$, we have*

$$\lim_{n \rightarrow \infty} P(n; \rho_n) = \begin{cases} 0 & \text{if } \lim_{n \rightarrow \infty} \alpha_n = -\infty \\ 1 & \text{if } \lim_{n \rightarrow \infty} \alpha_n = +\infty. \end{cases} \quad (54)$$

To the best of our knowledge there is no analog of Theorem 5.3 in the higher-dimensional case. The scaling ρ_F^{**} is also a strong critical scaling since asymptotically equivalent to ρ_F^* , i.e., $\rho_{F,n}^{**} \sim \rho_{F,n}^*$ as we note that

$$\frac{\rho_{F,n}^{**}}{\rho_{F,n}^*} = 1 - \frac{1}{r} \frac{\log \log n}{\log n}, \quad n = 2, 3, \dots \quad (55)$$

Theorem 5.3 is easily seen to imply the strong law (46)-(47). However, the converse is not true as the zero-one laws associated with the critical scalings ρ_F^* and ρ_F^{**} capture different levels of *sensitivity* to “small” deviations from criticality. For the same reasons that were given when discussing Theorem 4.3 in the uniform case, it is appropriate to interpret Theorem 5.3 as a very strong zero-one law in the non-uniform case, and to refer to the scaling ρ_F^{**} as a very strong critical scaling. It depends on the density f both through its minimum f_* and the parameter r which captures the smoothness of f near its minimum. Surprisingly enough, the “amplitude” value a makes no contribution!

When the density f achieves its minimum value f_* at *non-isolated* points (thereby violating Assumption 4), Theorem 5.3 needs to be modified as follows.

Theorem 5.4: *With Assumption 1 and Assumption 2 enforced on F , assume that $f(x) = f_*$ for all x in some non-empty open interval $I \subseteq (0, 1)$. Then, (54) still holds for any scaling $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ written in the form*

$$\rho_n = \frac{1}{f_*} \cdot \frac{1}{n} (\log n + \alpha_n), \quad n = 1, 2, \dots \quad (56)$$

with deviation function $\alpha : \mathbb{N}_0 \rightarrow \mathbb{R}$.

As expected we need only set $r = \infty$ in Theorem 5.3: Under the assumptions of Theorem 5.4, the density function f has

infinite smoothness near its infimum since locally flat there. Theorem 5.4 can also be viewed as an extension of Theorem 4.3 to distributions F whose density are locally constant (thus uniform) in a neighborhood of x_* .

D. Towards a shorter proof of Theorem 5.3

Theorem 5.3 was established in [22] by a variant of the method of first and second moments applied to the number of breakpoint users in $\mathbb{G}(n; \rho)$. Surprisingly, in the non-uniform case this approach turns out to be far more tedious to implement than any of the proofs given for Theorem 4.3 in [18], [21]. However, not all is lost: First, as pointed out earlier, Theorem 4.2 and Theorem 4.3 are easy consequences of (30) and (31), respectively. Next, Theorem 5.2 follows from (45) which is the analog of (30) for non-uniform distributions. Given that (31) complements (30), it is a small step to wonder whether (45) admits a similar complement, in which case such a result might form the basis for a short(er) proof of Theorem 5.3.

The form of the very strong critical scaling ρ_F^{**} suggests that a natural complement to (45) might take the following form.

Conjecture 5.5: *With Assumptions 3-5 enforced on the distribution F , we have*

$$nf_* M_n - \log n + \frac{1}{r} \log \log n + \gamma_n \Rightarrow_n \Lambda \quad (57)$$

where the sequence $\gamma : \mathbb{N}_0 \rightarrow \mathbb{R}$ depends on F and satisfies $\gamma_n = o(1)$.

Work is in progress on this conjecture; additional assumptions on F might be required. Earlier results by Deheuvels [4] point in the direction of Conjecture 5.5; see Section VI-D for details.

If Conjecture 5.5 were indeed correct, for each x in \mathbb{R} , we note that

$$\begin{aligned} &\mathbb{P} \left[nf_* M_n - \log n + \frac{1}{r} \log \log n + \gamma_n \leq x \right] \\ &= \mathbb{P} \left[M_n \leq \frac{\log n - \frac{1}{r} \log \log n - \gamma_n + x}{nf_*} \right] \\ &= \mathbb{P} \left[M_n \leq \rho_{F,n}^{**} + \frac{1}{f_*} \frac{x + o(1)}{n} \right] \end{aligned} \quad (58)$$

for all $n = 2, 3, \dots$. Using (57) we now get

$$\lim_{n \rightarrow \infty} P \left(n; \rho_{F,n}^{**} + \frac{1}{f_*} \frac{x}{n} \right) = e^{-e^{-x}} \quad (59)$$

with the help of an easy monotonicity argument. This convergence can be viewed as the analog of (32) for non-uniform distributions, and another monotonicity argument then leads readily from (59) to the conclusion of Theorem 5.3 – Indeed a short proof!

E. Phase transitions

As was the case for the uniform distribution, a convergence result such as (57) would allow us to characterize the width of phase transitions in the non-uniform case: Assume Assumptions 3-5 enforced on the distribution F . With obvious modifications to the notation, for each $n = 2, 3, \dots$ and each p in the interval $(0, 1)$, let $\rho_{F,n}(p)$ denote the unique solution

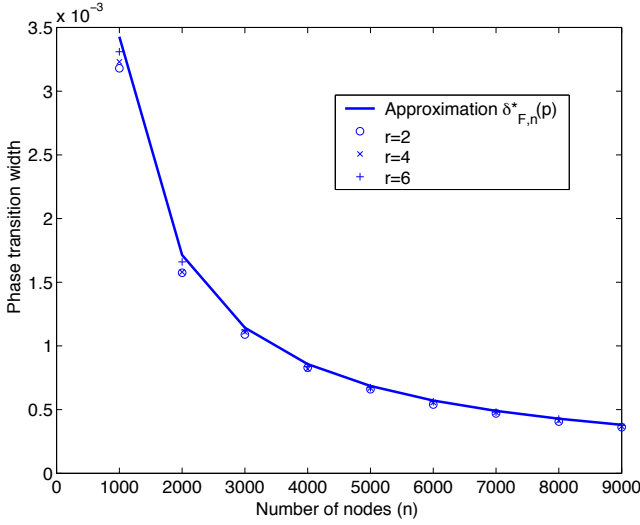


Fig. 2. Phase transition width when $p = 0.1$ and $f_r(x) = 0.9 + 0.1rx^{r-1}$ ($x \in [0, 1]$) with $r = 2, 4, 6$.

to (37). By arguments similar to the ones given for Theorem 4.4 [21], we readily obtain from Conjecture 5.5 (if valid) that

$$\rho_{F,n}(p) = \rho_{F,n}^{**} - \frac{1}{n} \log \left(\log \left(\frac{1}{p} \right) \right) + o(n^{-1}).$$

Thus, with constant $C(p)$ also given by (42), we conclude that

$$\begin{aligned} \delta_{F,n}(p) &= \rho_{F,n}(1-p) - \rho_{F,n}(p) \\ &= \frac{1}{f_*} \frac{C(p)}{n} + o(n^{-1}) \end{aligned} \quad (60)$$

for every p in the interval $(0, \frac{1}{2})$. It is worth noting that the impact of F on (the leading term in) the width transition is given only through f_* with the degree of smoothness r and the amplitude value a making no contribution at all! This remarkable lack of dependence is rather unexpected.

To illustrate this point, consider the density functions

$$f_r(x) = 0.9 + 0.1rx^{r-1}, \quad x \in [0, 1]$$

with $r = 2, 4, 6$. Taking (60) as our point of departure, we approximate the phase transition width through the quantity

$$\delta_{F,n}^*(p) := \frac{1}{f_*} \frac{C(p)}{n}, \quad n = 2, 3, \dots$$

for every p in the interval $(0, \frac{1}{2})$. This quantity is independent of the degree of smoothness r .

In Figure 2 we have displayed the simulation results together with the numerical approximations as n ranges from $n = 1000$ to $n = 9000$ in increments of 1000. The symbols represent the simulation results while the dash line gives the numerical approximation $\delta_{F,n}^*(p)$.⁶ For each value of n , it is clear that the transition widths for the three density functions are almost equal; as expected, the approximation accuracy improves as n becomes large. This provides some indirect confirmation of the validity of the Conjecture 5.5.

⁶Each symbol has been obtained as follows: For each value of n , an estimate of $P(n; \rho)$ was estimated by averaging the results of 10000 independent trials. This was done with ρ ranging over the unit interval with a very small granularity. Once this estimate becomes available, it is then possible to estimate the values of ρ at which the probability of connectivity is p and $1-p$, respectively, and calculate the transition width accordingly.

VI. DISCUSSION

A. Uniform vs. non-uniform

Under the assumptions of Theorem 5.2, the comparison

$$\rho_{U,n}^* \leq \rho_{F,n}^*, \quad n = 1, 2, \dots \quad (61)$$

holds since $f_* \leq 1$, showing that the uniform distribution yields the *smallest* strong critical scaling in the class of distributions satisfying Assumptions 1 and 2.

The value of f_* is typically not known to the network users, and there seems to be little operational reason for them to have this knowledge (especially when nodes are mobile). Since f_* is the minimum of a density function, estimating it will be fraught with difficulties akin to those encountered in the estimation of probabilities of rare events. In particular, the unavailability of data sets large enough could lead to poor estimates.

Under these circumstances the sensitivity to deviations represented by the strong zero-one laws of Theorem 5.2 and Theorem 5.3 *cannot* be leveraged in any meaningful way to guide the power allocation at the nodes: The sharp phase transitions discussed in earlier sections, though theoretically pleasing, cannot be exploited practically as this would require not only knowledge of f_* but also the availability of the smoothness parameter r . In practice we are left with weak zero-one laws as we note that the scaling ρ_U^* is a weak critical scaling, a *robust*, albeit weak, conclusion which holds across *all* distributions F satisfying (16).

B. From uniform to non-uniform node placement

Earlier we already remarked that $\rho_F^* = \rho_U^*$ when F is the uniform distribution. Thus, as we pass from Theorem 4.2 to Theorem 4.3, it might have been tempting to infer from the strong zero-one law (46)-(47) that in the non-uniform case the very strong zero-one law would be valid for scalings $\rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ written in the form

$$\rho_n = \frac{1}{f_*} \cdot \frac{1}{n} (\log n + \beta_n), \quad n = 1, 2, \dots \quad (62)$$

with deviation function $\beta : \mathbb{N}_0 \rightarrow \mathbb{R}$. Under this guess, the strong critical scaling ρ_F^* would also have been a very strong critical scaling.

Under the assumptions of Theorem 5.3 this guess is in fact incorrect with the following consequences: For instance, the scaling $\tilde{\rho} : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ given by

$$\tilde{\rho}_n = \frac{1}{f_*} \cdot \frac{1}{n} \left(\log n - \frac{1}{2r} \log \log n \right), \quad n = 1, 2, \dots \quad (63)$$

is of the form (62) with deviations $\beta_n = -\frac{1}{2r} \log \log n$ for all $n = 1, 2, \dots$. Were our guess correct, we would conclude erroneously that $\lim_{n \rightarrow \infty} P(n; \tilde{\rho}_n) = 0$. Instead Theorem 5.3 yields the correct conclusion $\lim_{n \rightarrow \infty} P(n; \tilde{\rho}_n) = 1$ since the scaling $\tilde{\rho}$ is also of the form (52) with $\alpha_n = \frac{1}{2r} \log \log n$ for all $n = 1, 2, \dots$. Thus, in the framework of Theorem 5.3, the extreme sensitivity to deviations expressed by a very strong zero-one law, is now given in terms of deviations taken relative to ρ_F^{**} (and not to ρ_F^*). This change in baseline is remarkable in light of the fact that $\rho_{F,n}^*$ and $\rho_{F,n}^{**}$ become very quickly indistinguishable from each other as n increases! Indeed, the

very fast convergence in $\lim_{n \rightarrow \infty} (\rho_{F,n}^* - \rho_{F,n}^{**}) = 0$ is an immediate consequence of the observation

$$\rho_{F,n}^* - \rho_{F,n}^{**} = \frac{1}{r} \frac{\log \log n}{n}, \quad n = 2, 3, \dots$$

On the other hand, under the assumptions of Theorem 5.4 the guess based on (63) is the correct one, since ρ_F^* is also a very strong scaling in that setting.

C. The smoother, the larger

Consider now two distributions F_1 and F_2 satisfying the conditions of Theorem 5.3 with parameters $(f_{1,*}, r_1)$ and $(f_{2,*}, r_2)$. If $f_{1,*} = f_{2,*}$, the comparison

$$\rho_{F_1,n}^{**} \leq \rho_{F_2,n}^{**}, \quad n = 1, 2, \dots \quad (64)$$

holds whenever

$$r_1 \leq r_2. \quad (65)$$

Thus, the smoother the density f at x_* , the larger the very strong critical scaling.

D. Conjecture 5.5 and earlier results by Deheuvels

In the context of Conjecture 5.5 it is appropriate to mention some earlier results by Deheuvels [4]. They are given under the following conditions somewhat reminiscent of Assumptions 3-5: (i) The density function f is continuous on $(0, 1)$; (ii) The minimizer x_* appearing in (16) is assumed to be an *isolated* minimizer; (iii) For some finite constant $r > 0$, we have $0 < d_r \leq D_r < \infty$ where⁷

$$d_r := \liminf_{h \rightarrow 0} \left(\frac{f(x_* + h) - f(x_*)}{|h|^r} \right)$$

and

$$D_r := \limsup_{h \rightarrow 0} \left(\frac{f(x_* + h) - f(x_*)}{|h|^r} \right).$$

Under these conditions, Deheuvels [4, Thm. 4, p. 1183] (where $k = 1$) has shown that

$$\liminf_{n \rightarrow \infty} \left(\frac{n f_* M_n - \log n}{\log \log n} \right) = -\frac{1}{r} \quad a.s. \quad (66)$$

and

$$\limsup_{n \rightarrow \infty} \left(\frac{n f_* M_n - \log n}{\log \log n} \right) = 2 - \frac{1}{r} \quad a.s. \quad (67)$$

These results certainly point in the direction of the conjectured convergence (57).

As we recall that convergence in distribution is equivalent to convergence in probability when the limit is a.s. constant, we conclude from (57) that

$$\frac{n f_* M_n - \log n}{\log \log n} \xrightarrow{P} n - \frac{1}{r}, \quad (68)$$

but this does not contradict (66)-(67) as these convergence statements are given in the stronger a.s. sense.

⁷This is the form that the conditions take when x_* is an interior point of the interval $[0, 1]$. Obvious modifications need to be made when either $x_* = 0$ or $x_* = 1$.

VII. VANISHING DENSITIES

A natural question arises as to the validity and form of the results of Section V when the density f vanishes on the interval $[0, 1]$.

A. A weak zero-one law

When $f_* = 0$, a blind substitution in (44) yields $\rho_{F,n}^* = \infty$ for all $n = 1, 2, \dots$, and this begs the question as to what is the appropriate analog of Theorem 5.2. No general answer to this question is available given that it is shaped in a crucial way by the properties of the density where it vanishes.

In [19] we have shown through simple examples that when (16) fails, the property of graph connectivity exhibits only a weak zero-one law: More specifically, with $p > 0$ consider the probability distribution F_p given by

$$F_p(x) = x^{p+1}, \quad x \in [0, 1] \quad (69)$$

with corresponding density function f_p given by

$$f_p(x) = (p+1)x^p, \quad x \in [0, 1]. \quad (70)$$

Theorem 5.2 is now replaced by the following result.

Theorem 7.1: Assume F to be given by (69) for some $p > 0$. The property of graph connectivity in the random graph $\mathbb{G}(n; \rho)$ admits only a weak zero-one law, and the scaling $\rho_p^ : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ given by*

$$\rho_{p,n}^* = n^{-\frac{1}{p+1}}, \quad n = 1, 2, \dots \quad (71)$$

is the corresponding weak critical scaling.

To get a sense as to why this is so, we refer the reader to the discussion in [19] where we provided elementary arguments to show that

$$\frac{M_n}{\rho_{p,n}^*} \xRightarrow{n} L_p \quad (72)$$

for some non-degenerate rv L_p with $0 < L_p < \infty$ a.s. As (24) fails (since L_p is non-degenerate), Proposition 3.1 precludes the existence of a strong zero-one law. The existence of a weak zero-one law now follows readily from (25) and (72); see [19] for details.

B. Discussion

As mentioned earlier, the scaling ρ_U^* is a weak critical scaling under *all* distributions F satisfying (16). However, with F given by (69), the critical scaling given by (71) is now of a much larger order since

$$\frac{\log n}{n} = o\left(n^{-\frac{1}{p+1}}\right).$$

Implications for resource dimensioning take the following form: Critical scalings serve as proxy for the critical transmission range when n is large. Thus, under node placement with a vanishing density such as (69), the critical transmission range is orders of magnitude *larger* than would otherwise have been the case when (16) holds, resulting in *higher* minimum power levels to ensure connectivity. Similar *qualitative* conclusions were already pointed out by Santi [36, Thm. 4] for two-dimensional networks under the random waypoint mobility

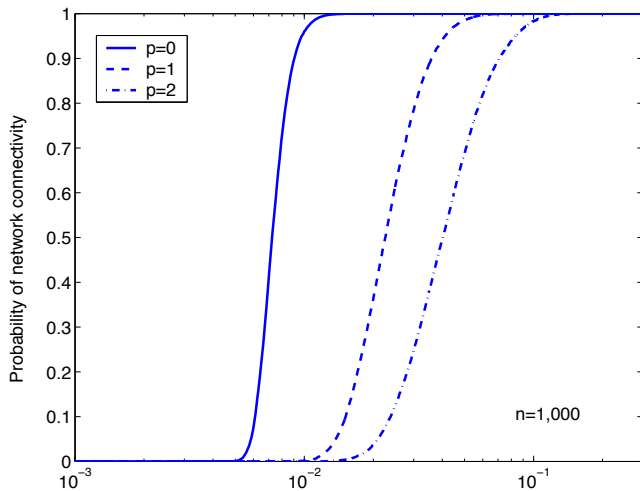


Fig. 3. Phase transition with $n = 1,000$

model without pause. In one dimension, the corresponding stationary spatial node density is given by

$$f_{\text{RWP}}(x) = 6x(1-x), \quad 0 \leq x \leq 1. \quad (73)$$

Here, under (69) we can go beyond qualitative statements and give *precise* information on the *order* of the asymptotics for the critical transmission range.

Although the distribution (69) was selected because its simpler form facilitated the analysis, it is nevertheless representative of vanishing densities such as (73). Indeed, both Theorems 5.2 and 7.1 derive from limiting properties of the maximal spacing under F . Such properties are influenced by the behavior of the density in the vicinity of its minimum point [23, p. 519]: The densities (70) (with $p = 1$) and (73) have similar behavior near $x = 0$ since $f_{\text{RWP}}(x) \sim 6x$ as $x \simeq 0$. Thus, the results discussed here suggest that this model requires a much larger critical transmission scaling given by

$$\rho_{\text{RWP},n}^* = \frac{1}{\sqrt[n]{n}}, \quad n = 1, 2, \dots$$

Under uniform node placement, the convergence (32) crisply captures the fact that the phase transition associated with very strong zero-one laws is very sharp indeed [15], [17], [21]. In the non-uniform case with $f_* > 0$ the conjectured convergence (57) plays a similar role. However, the absence of (very) strong critical scalings under (69) precludes such convergence, and essentially rules out the possibility that the corresponding phase transition will be sharp in this case.

These conclusions are already apparent from the limited simulation results displayed in Figure 3 where nodes are placed on $[0, 1]$ according to F_p with $p = 0, 1, 2$; the case $p = 0$ corresponds to the uniform distribution. For each $p = 0, 1, 2$, the figure displays the corresponding plot of $P(n, \rho)$ as a function of ρ (in base 10 log-scale) for $n = 1,000$. As expected, the phase transition is much sharper for $p = 0$ than for positive p . These displays also suggest that the sharpness of the phase transition decreases with increasing p . However, at the time of this writing, we are not in a position to offer precise *quantitative* results validating this claim.

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